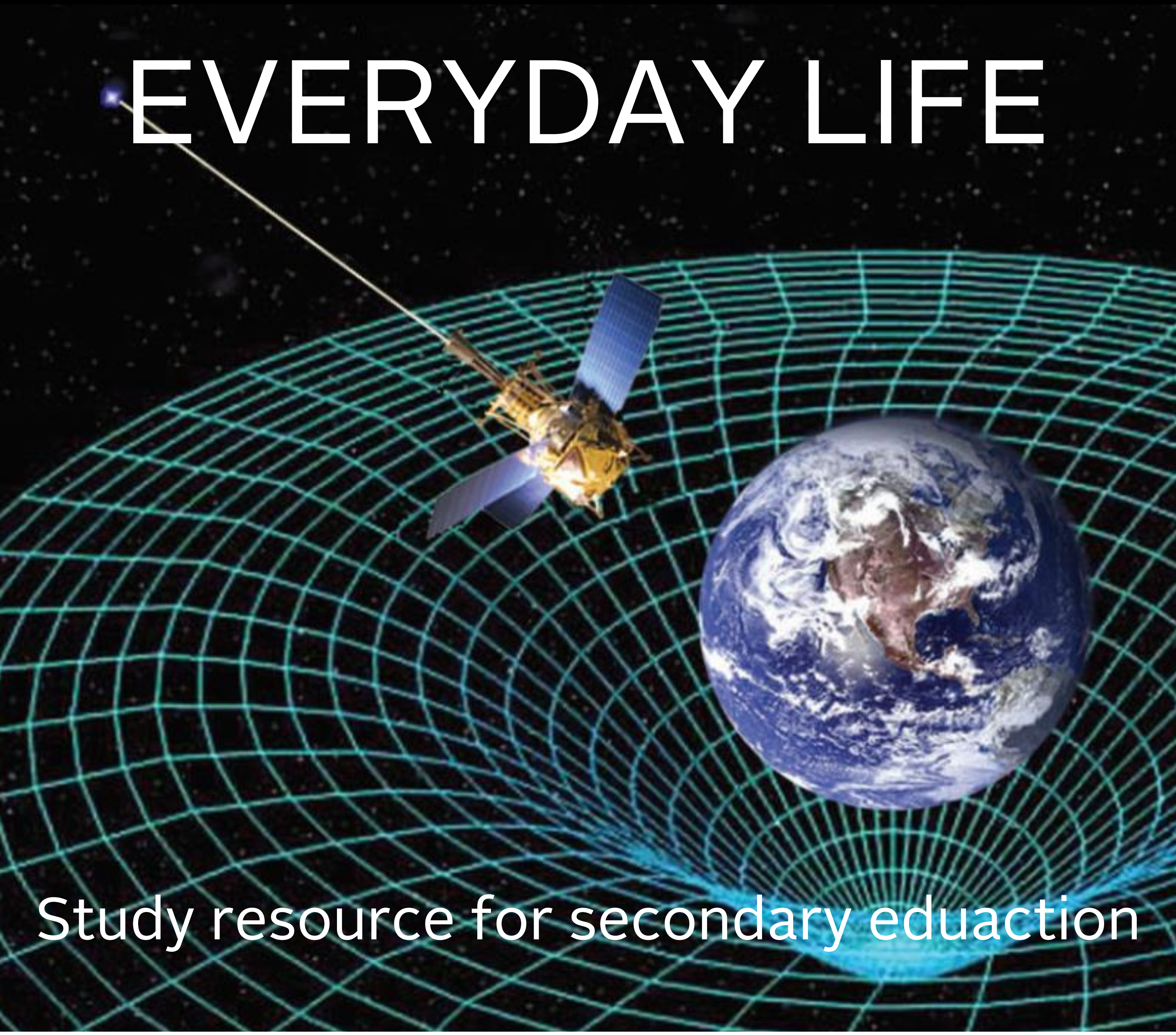
STUDENT BOOKLET
Fysikk 2 & English

EINSTEINIAN RELATIVITY and EVERYDAY LIFE



Study resource for secondary eduaction

Nordic esero

NAROM | ANDØYA SPACE CENTER

Brief description

This booklet gives an overview of how relativistic effects affects GPS-satellites.

Learning outcomes

The student will get some experience with simple calculations in special and general relativity, as well as some exposure to the mind-bending principles and consequences of these theories.

Contents

What good is fundamental physics to the person on the street?

This is the perennial question posed to physicists by their non-science friends, by students in the humanities and social sciences, and by politicians looking to justify spending tax dollars on some fundamental research project. It is hard to predict with certainty what the return on investment in fundamental physics research will be, though few dispute that physics is somehow 'good'.

In response to this, physicists have become adept at finding good examples of the long-term benefit of fundamental research: the quantum theory of solids led to semiconductors and computer chips, nuclear magnetic resonance led to MRI imaging, and particle accelerators led to beams for cancer treatment. However, what about Einstein's theories of special and general relativity? One could hardly imagine a branch of fundamental physics less likely to have practical consequences. But surprisingly enough, relativity plays a key role in a multi-billion dollar growth industry centred around the Global Positioning System (GPS).

When Einstein finalized his theory of gravity and curved space-time in November 1915, ending a quest that he began with his 1905 special relativity, he had little concern for practical or observable consequences. He was unimpressed when measurements of the bending of starlight in 1919 confirmed his theory. Even today, general relativity plays its main role in the astronomical domain, with its black holes, gravity waves and the cosmic big bang – or in the domain of the ultra-small, where theorists look to unify general relativity with the other fundamental interactions, using exotic concepts such as strings and branes.

But GPS is an exception. Originally built at a cost of over $10 billion mainly for military navigation, GPS has rapidly transformed itself into a thriving commercial industry. The system is based on an array of 24 satellites orbiting the earth, each carrying a precise atomic clock. Using a hand-held GPS receiver which detects radio emissions from any of the satellites which happen to be overhead, users of even moderately priced devices can determine latitude, longitude and altitude to an accuracy which can currently reach 15 meters, and local time to 50 billionths of a second. Apart from the obvious military uses, GPS is finding applications in airplane navigation, oil exploration, wilderness recreation, bridge construction, sailing, and interstate trucking, to name just a few.

However, in a relativistic world, things are not so simple. The satellite clocks are moving at 14 000 km/h in orbits that circle the Earth twice per day, much faster than clocks on the surface of the Earth, and Einstein's theory of special relativity says that rapidly moving clocks tick more slowly, in this case by about seven microseconds (millionths of a second) per day.

In addition, the orbiting clocks are 20 000 km above the Earth, and experience gravity that is four times weaker than that on the ground. Einstein's general relativity theory says that gravity curves space and time, resulting in a tendency for the orbiting clocks to tick slightly faster, by about 45 microseconds per day. The net result is that time on a GPS satellite clock advances *faster* than a clock on the ground by about 38 microseconds per day.

To determine its location, a GPS receiver uses the time at which the individual signals from each satellite was emitted – as determined by the atomic clock on-board the satellite and encoded into the signal – together with the speed of light, to calculate the distance between itself and the satellites it communicated with. The orbit of each satellite is known accurately. Given enough satellites, it is a simple geometric problem to compute the receiver's precise location, both in space and time. To achieve a navigation accuracy of 15 meters, time throughout the GPS system must be known to an accuracy of 50 nanoseconds, which simply corresponds to the time required for light to travel 15 meters.

But at 38 microseconds per day, the relativistic offset in the rates of the satellite clocks is so large that, if left uncompensated, it would cause navigational errors that accumulate to more than 10 km per day! GPS accounts for relativity by electronically adjusting the rates of the satellite clocks, and by building mathematical corrections into the receiver chips that solve for the user's location. Without the proper application of relativity, GPS would fail in its navigational function within about 2 minutes.

So, the next time your plane approaches an airport in bad weather, and you just happen to be wondering "what good is fundamental physics?" think about Einstein and the GPS tracker in the cockpit, helping the pilots guide you to a safe landing.

Calculating the errors introduced in GPS because of relativistic effects

Task 1 – Special relativity

Moving clocks tic slower than stationary ones.[1] This effect is known as (*velocity*) *time dilation*, and is given by

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{1}$$

where $\Delta t'$ is the time interval measured in the moving system (e.g. by the atomic clock on the satellite), Δt is the time interval measured in the stationary reference system (e.g. on Earth), v is the relative (satellite) velocity, and $c \approx 3.000 \times 10^8\ \mathrm{m/s}$ is the speed of light in vacuum.

a) Assume that the velocity of the GPS-satellite with respect to the surface of the Earth is 3874 m/s (around 14 000 km/h) and verify that – as seen from Earth – the satellite clock indeed ticks about 7 μs/day slower than an atomic clock on the surface.

b) Calculate the absolute error in localization per day (24 hours, as measured with an atomic clock on the surface of the Earth) due to special relativity. (Tip: The direction of the localization error can be ignored, as can its evolution – we are simply looking for a worst-case estimate).

c) Repeat the calculations in (a) and (b), this time from the perspective of the satellite. (A 'day' should still be 24 hours, not the time it takes the satellite to orbit the Earth). How does the results compare?

[1] Note that what counts as 'stationary' and 'moving' are entirely observer dependent. In this exercise, we will consider the surface of the Earth to be unmoving, as it turns out the general relativity calculations are easier that way.

Task 2 – General relativity

Clocks closer to a massive object (for example the Earth) tic slower than clocks further away (e.g. satellites), an effect known as *gravitational time dilation*. More specifically, the time dilation on a GPS-satellite due *only* to the gravitational field of Earth is:[2]

$$\frac{\Delta t'}{\Delta t} \approx \sqrt{1 - \frac{2GM_E}{c^2}\left(\frac{1}{R_{GPS}} - \frac{1}{R_E}\right)} \qquad (2)$$

where Δt, $\Delta t'$ and c is as in equation (1), $G \approx 6.674 \times 10^{-11}\ \mathrm{m^3 kg^{-1} s^{-2}}$ is the gravitational constant, $M_E \approx 5.972 \times 10^{24}$ kg is the mass of the Earth, $R_E \approx 6371$ km is the radius of the Earth and $R_{GPS} \approx 20\,200\ \mathrm{km} + R_E$ is the radial distance of the GPS from the centre of the Earth.

a) Verify that the gravitational time dilation causes the atomic clock on the satellite to tick about 45 μs faster than an atomic clock on the surface of the Earth, as claimed in the introduction.

b) Calculate the absolute error in localization per day (24 hours, as measured with an atomic clock on the surface of the Earth) due to gravitational effects.

c) Verify the final numbers in the text by calculating the total difference between the clock tics (GPS-satellite vs the surface of the Earth), and the effective absolute mistake in localization per day.

d) How have engineers solved this problem?

Special relativity gets its name because it is a special case of general relativity (namely, the one where we ignore the effects of gravity). Hence, the effects of special relativity are naturally included in general relativity. Moreover, it turns out that the effects of special relativity cannot simply be added with the effects of gravity to obtain the combined effect of gravity and velocity like we have done here. Instead, it turns out that the proper way to combine the effects of (1) and (2) is

$$\frac{\Delta t'}{\Delta t} = \sqrt{\frac{1 - \frac{r_s}{R_{GPS}} - \frac{v^2}{c^2}}{1 - \frac{r_s}{R_E}}}, \qquad r_s \equiv \frac{2GM_E}{c^2} \qquad (3)$$

[2] Strictly speaking, the effects of movement and gravity cannot be separated out like we do here when they occur simultaneously. However, as we will see, in *this* context the error is negligible.

(this equation cannot be obtained from (1) and (2) – instead it follows from the same derivation as was done to obtain (2) if one includes an angular velocity $\mathrm{d}\theta/\mathrm{d}t = v/R_{GPS}$), where we for simplicity introduced $r_s = 2GM_E/c^2$, known as the *Schwarzschild radius* of Earth.[3]

e) Repeat the calculations in (c), this time using the more correct expression given in (3). Comments?

Further study

The attentive reader might suspect the approximation in equation (2) to be of importance. To the inquisitive reader, the exact expression for the time dilation due *only* to the gravitational field of the Earth is given here:

$$\frac{\Delta t'}{\Delta t} = \sqrt{\frac{1 - \frac{r_s}{R_{GPS}}}{1 - \frac{r_s}{R_E}}}, \qquad r_s \equiv \frac{2GM_E}{c^2} \tag{4}$$

It is left as an exercise to the reader to verify that (2) indeed follows from (4),[4] and to explore the consequences of employing a similar approximation in equation (3).

Other resources

The interested reader is hereby referred to PBS Spacetime's excellent YouTube playlist covering general relativity:

https://www.youtube.com/playlist?list=PLsPUh22kYmNAmjsHke4pd8S9z6m_hVRur

Don't despair if the videos prove difficult to understand – this is challenging material, and well beyond the curriculum in Fysikk 2!

Note

This document is developed by NAROM for Nordic ESERO.

[3] Any object smaller than its Schwarzschild radius becomes a black hole. It is left as an exercise to the reader to calculate the radius at which the Earth becomes a black hole.

[4] Hint: Start by adding 0 in such a way that the main fraction can be split into two parts, one of which is just equal to 1!